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Stochastic iterative method for a system of parabolic equations

Vorticity form of the linearized Navier–Stokes equations is considered. Stochastic iterative methods for calculation of the velocity field based on the potential representation of the vorticity are constructed.

1. Introduction

It is well known that one of the efficient algorithms when solving the exterior flows problems is the random vortex method. Its iterative scheme proposed by Chorin [2] is based on the supposition that the vorticity field can be sufficiently approximated by a finite sum of particles carrying vorticity, these particles being named vortices.

At every time–step the velocity field is supposed to be fixed first. Therefore, the linearized vorticity form of three-dimensional Navier–Stokes equations

$$w_t = \nu \Delta w - u \cdot \nabla w + w \cdot \nabla u \tag{1}$$

can be considered as the Kolmogorov equation for stochastic ordinary differential equation in $\mathbb{R}^3$ describing the diffusion and convection of vortices.

A step of the random vortex algorithm is ordinarily based on some explicit computational scheme, Euler or Runge–Kutta type. Next, new values of the velocity field generated by the vortex particles are computed in accordance with the Biot–Savart formula. Thus, by such interaction of stochastically moving particles [1] the nonlinearity of the Navier–Stokes equations and stretching of the vorticity are taken into account.

Note, that a computational implementation of the random vortex method involves three sources of a discretization error: approximation of w error, error due to the discretization in time variable and error of approximation of the kernel in the Biot–Savart formula.

In this paper we propose a pure stochastic computational algorithm for numerical calculation of the velocity induced by the vorticity field. The velocity is considered to be a linear functional of the vorticity being a solution to the initial value problem

$$w(x,0) = w_0(x),\ x \in \mathbb{R}^3 \tag{2}$$

for (1).

Suppose that $u(x,t)$ is a given bounded incompressible field, deterministic to start with. Then (1) can be considered as a system of linear parabolic equations. Suppose that u_i, w_i, $\frac{\partial u_i}{\partial x_j}$, $\frac{\partial w_i}{\partial x_j}$, $i,j = 1,2,3$ are bounded and continuous in $D = \mathbb{R}^3 \times (0,T]$ for some finite T. With these restrictions, the Cauchy problem (1), (2) is equivalent to the integral–differential equation (potential representation)

$$w = \int_0^t \int_{\mathbb{R}^3} (-u \cdot \nabla + w \cdot \nabla u) Z + \int_{\mathbb{R}^3} w_0 Z,$$

$Z(x - x', t - t')$ being the fundamental solution to the heat equation.

Let w be divergence free (remind that w is supposed to be not depending on u). Then the Ostrogradsky formula implies that w satisfies the system of integral equations

$$w_i = \int_0^t \int_{\mathbb{R}^3} (w_i u - u_i w) \cdot \nabla Z + \int_{\mathbb{R}^3} w_{0i} Z,\ i = 1,2,3,$$

or

$$w(x,t) = \int_0^t dt' \int_{\mathbb{R}^3} dx' K(x,t;x',t') w(x',t') + F_0(x,t), \tag{3}$$

where $K = \{k_{ij}\}_{i,j=1}^3$ is the matrix of weakly singular kernels. (Note, that for $\dfrac{\partial u_i}{\partial x_j}$ being locally Hölder continuous with respect to x uniformly in t (3) is equivalent in $C^{2,1}$ to the Cauchy problem (1), (2) [3]). Denote by $\mathcal{K}$ the integral operator of this system. Then (see, e.g., [4], [7]) by induction the inequality

$$|(\mathcal{K}^n F_0)_i(x,t)| \leq \frac{(3A\pi^{1/2}t^{1/2})^n}{\Gamma(1+n/2)} ||F_0||_{L_\infty(D)},\ i = 1,2,3,$$

is valid, A being a constant depending on the upper bound for $|u|$. Therefore, by the asymptotic properties of Γ–function the Neumann series for (3) converges uniformly in D. Furthemore, unbiased Monte Carlo estimators for w and integral functionals of w can be constructed [6].

2. Estimators for velocity: adjoint estimator

The Biot–Savart formula

$$u_{\text{new}}(x,t) = \int_{\mathbb{R}^3} dx' H(x,t;x',t') w(x',t'), \tag{4}$$

where

$$H = -\frac{1}{4\pi|x-x'|^3} \begin{pmatrix} 0 & -(x_3 - x_3') & x_2 - x_2' \\ x_3 - x_3' & 0 & -(x_1 - x_1') \\ -(x_2 - x_2') & x_1 - x_1' & 0 \end{pmatrix},$$

yields that, given bounded u and w_0, standard Monte Carlo estimators can be employed to compute new value of the velocity at a point (x,t), $t > 0$.

Consider an adjoint estimator for the integral functional (4) first. To construct it, one has to determine a particular Markov chain $\{(x_n, t_n) \in D, n = 0,1,\ldots\}$ with the disribution density of the initial point p_0 and the transition density $p(x,t \to x't')$ consistent with H and $K(x,t;x',t')$, respectively. It is clear that p can be taken to be equal to

$$q\,\frac{\nu^{1/2}}{2\pi t^{1/2}} \frac{|x-x'|}{(4\nu(t-t'))^{5/2}} \exp\left(-\frac{|x-x'|^2}{4\nu(t-t')}\right),\ \text{for } t > t' > 0,$$

and zero in all other cases. Here q is the survival probability.

This means that $x_{n+1} = x_n + 2(\nu(t_n - t_{n+1})\gamma_2^{(n)})^{1/2}\omega^{(n)}$, $t_{n+1} = (1-\alpha^2)t_n$, where α is uniformly distributed in $(0,1)$, $\omega^{(n)}$ is a unit isotropic vector and $\gamma_2^{(n)}$ is Γ–distributed with the parameter 2.

As for the initial density, to take into account the singularity of the kernel H, suppose that $u_{\text{new}}(x,t)$, (x,t) given, is to be calculated. So, one can choose some positive finite R_0, α_0 and sample x_0 with probability α_0 in the ball $B(x,R_0) = \{x' : |x-x'| < R_0\}$ in accordance with the density $p_{01} = (4\pi R_0|x-x'|^2)^{-1}$. With the complementary probability, x_0 is sampled in $\mathbb{R}^3 \setminus B(x,R_0)$ in accordance with some density that takes into account the asymptotic properties of the product Hw.

Hence, the Markov chain constructed, the adjoint collision estimator looks like as follows,

$$\xi^*(x,t) = \sum_{n=0}^{N} Q_n^* w_0(x_n^*). \tag{5}$$

Here N is a random number of the last point of the path, x_n^* is of the Gaussian distribution $N(x_n, 2\nu t_n)$, and $Q_{n+1}^* = \dfrac{K(x_n,t_n;x_{n+1},t_{n+1})}{p(x_n,t_n \to x_{n+1},t_{n+1})}\,Q_n^* = A_n^* Q_n^*$ are random weight matrices,

$$(A_n^*)_{ij} = \frac{4}{q}\left(\frac{t_n}{\pi\nu}\right)^{1/2}\left[\left(-\sum_{k=1}^{3} u_k\omega_k^{(n)}\right)\delta_{ij} + u_i\omega_j^{(n)}\right], \tag{6}$$

where values of the velocity u are taken at the point (x_{n+1}, t_{n+1}). By the appropriate choice of the survival probability one can obtain the uniform boundedness of the weight factors A_n^* thus ensuring that the estimator (5) is unbiased:

$$\mathbb{E}\xi^*(x,t) = u_{\text{new}}(x,t)$$

and its variance is bounded.

3. Direct estimator

Consider now a direct estimator for the functional (4) of a solution to the integral equation (3). According to the general theory of the Monte Carlo method one has to construct a Markov chain $\{(x'_n, t'_n) \in D, n = 0, 1, \ldots\}$ determined by an initial distribution consistent with F_0 and a transition density $p_1(x'_n, t'_n \to x'_{n+1}, t'_{n+1})$ consistent with $K(x'_{n+1}, t'_{n+1}; x'_n, t'_n)$. We choose p_1 to be equal to

$$q\,\frac{\nu^{1/2}}{2\pi(t-t'_n)^{1/2}}\,\frac{|x'_n - x'_{n+1}|}{\left(4\nu(t'_{n+1}-t'_n)\right)^{5/2}}\exp\left(-\frac{|x'_n - x'_{n+1}|^2}{4\nu(t'_{n+1}-t'_n)}\right),$$

for $t > t'_{n+1} > t'_n > 0$, and zero in all other cases. Hence we have $x'_{n+1} = x'_n + 2(\nu(t'_{n+1} - t'_n)\gamma_2^{(n)})^{1/2}\omega^{(n)}$, $t'_{n+1} = t'_n + (t - t'_n)\alpha^2$.

In the full analogy with the adjoint case, for an appropriate choice of the survival probability the direct collision estimator

$$\xi(x,t) = \sum_{n=0}^{N} H(x, t; x'_n, t'_n) Q_n, \tag{7}$$

will be unbiased. Here $Q_0 = \dfrac{w_0(x'^*_0)}{p_{01}(x'_0, t'_0)}$, $Q_{n+1} = A_n Q_n$,

$$(A_n)_{ij} = \frac{4}{q}\left(\frac{t-t'_n}{\pi\nu}\right)^{1/2}\left[\left(-\sum_{k=1}^{3} u_k\omega_k^{(n)}\right)\delta_{ij} + u_i\omega_j^{(n)}\right],$$

values of the velocity u are taken at the point (x'_n, t'_n); x'^*_0 is of the Gaussian distribution $N(x'_0, 2\nu t'_0)$.

To ensure that the variance of (7) is bounded, one has to take into account the singularity of H. This can be achieved by including it into the initial density p_{01}.

Note that both Markov chains $\{(x_n, t_n)\}$ and $\{(x'_n, t'_n)\}$ used here are non-diffusion stochastic sequences.

Suppose now that the initial vorticity is approximated by a sum of point vortices:

$$w_0(x) = \sum_{k=1}^{m} w_0^{(k)}\delta(x - x^{(k)}).$$

Then we have

$$F_0(x,t) = \sum_{k=1}^{m} w_0^{(k)} Z(x - x^{(k)}, t),$$

and it seems to be natural to take $p_{01}^{(k)}(x'_0, t'_0) = \dfrac{1}{t} Z(x'_0 - x^{(k)}, t'_0)$, $k = 1, 2, \ldots, m$ as the consistent with $F_0(x,t)$ initial densities, sample m independent initial points $(x'^{(k)}_0, t'^{(k)}_0)$: $t'^{(k)}_0 = \alpha t$ and $x'^{(k)}_0$ being of the Gaussian distribution $N(x^{(k)}, 2\nu t'^{(k)}_0)$, and set the estimator for $u_{\text{new}}(x,t)$ to be the sum of the estimators (7) for different k. $Q_0^{(k)}$ being equal to $t\, w_0^{(k)}$ here. (Note, that this is possible due to the linearity of the integral operator $\mathcal{K}$).

It is essential to note, however, that the procedure described can lead to the estimator with the unbounded variance, since the singularity of the weight function H is not taken into account. So, one encounters with the same problem as when constructing the random vortex algorithm. This problem can be resolved in the analogous way, namely by cut-off of the function $Z(x'_0 - x^{(k)}, t'_0)H(x^{(k)}, t; x'_n, t'_n)$ in the vicinity of the point $x^{(k)}$ where the velocity is to be computed.

4. Numerical experiments

The numerical experiments have been carried out for the particular case of the Stokes viscous flow in the domain G exterior to the unit sphere [5]. So, the velocity field is taken to be $u_1 = 1-0.75/|x|-0.25/|x|^3+0.75(1/|x|^2-1)x_1^2/|x|^3$, $u_2 = 0.75(1/|x|^2-1)x_1x_2/|x|^3$, $u_3 = 0.75(1/|x|^2-1)x_1x_3/|x|^3$, the initial vorticity is equal to $w_{01} = 0$, $w_{02} = 1.5x_3/|x|^3$, $w_{03} = -1.5x_2/|x|^3$ for $x \in G = \{x : |x| > 1\}$. F_0 is set to be equal to zero in $\mathbb{R}^3 \setminus G$.

The adjoint estimator (5) has been employed in order to calculate values of the velocity $u_{\mathrm{new}}(x,t)$ at the points $x^{(k)} = k\,(\sqrt{2}, \sqrt{2}, 0),\ k = 0, 1, \ldots, 10,\ t$ being equal to 0.1.

The following results were obtained

k	u_1	u_2	$u_{\mathrm{new},1}$	$u_{\mathrm{new},2}$
0	0.000	0.000	0.16	-0.00
1	0.071	-0.059	0.20	-0.02
2	0.135	-0.096	0.23	-0.03
3	0.191	-0.118	0.26	-0.06
4	0.242	-0.131	0.30	-0.07
5	0.287	-0.139	0.32	-0.07
6	0.327	-0.143	0.36	-0.07
7	0.364	-0.144	0.38	-0.09
8	0.396	-0.144	0.41	-0.10
9	0.426	-0.143	0.44	-0.12
10	0.453	-0.141	0.46	-0.11

with $\sigma = 0.008$ for $u_{\mathrm{new},1}$ and 0.006 for $u_{\mathrm{new},2}$.

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5. References

1 Bossy, M., Talay, D.: A stochastic particle method for the McKean-Vlasov and the Burgers equation; Math. Comp. **66** (1997), 157–192.
2 Chorin, A.J.: Numerical study of slightly viscous flow; J. Fluid Mech. **57** (1973), 785–796.
3 Friedman, A.: Partial differential equations of parabolic type; Prentice-Hall, 1964.
4 Il'in, A.M., Kalashnikov, A.S., Olejnik, O.A.: Linear second order equations of parabolic type; Uspekhi Mat. Nauk **17** (1962), Issue 3, 3–147 (in Russian).
5 Landau, L.D., Lifshitz, E.M.: Theoretical physics. Vol.6. Hydrodynamics; Nauka, Moscow, 1988 (in Russian).
6 Mikhailov, G.A.: Optimization of weighted Monte Carlo methods; Springer-Verlag, 1992.
7 Simonov, N.A.: Stochastic iterative methods of solving equations of parabolic type; Sibirskii Matematicheskii Journal (1997), to appear (in Russian).

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