Stochastic computational methods for parabolic equations with random data

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ABSTRACT

Cauchy problem for a parabolic equation with random source and initial data is considered. Monte Carlo estimators for the mean solution and its covariance function are constructed on the basis of the potential representations.

INTEGRAL FORMULATION

Consider a random parabolic equation

$$w_t = \nu \Delta w - u \cdot \nabla w + f \tag{1}$$

(or $Hw = f$) in $Q \equiv \mathbb{R}^m \times (0, T]$ and the Cauchy problem for it

$$w(x, 0; \omega) = w_0(x; \omega), x \in \mathbb{R}^m. \tag{2}$$

Here $f(x, t; \omega), w_0(x; \omega)$ are random with uniformly bounded second moments and continuous sample functions, $\omega \in (\Omega, \mathcal{A}, P)$, some suitable probability space. Equations (1), (2) are to be satisfied with probability one.

Assume that f is mean square continuous in $\overline{Q}$ and mean square Hölder continuous in x uniformly in $\overline{Q}$, w_0 is mean square continuous in $\mathbb{R}^m$. Then [1], the unique solution to the problem (1), (2) in $L_2(\Omega)$ exists.

Suppose that our aim is to calculate the mean value $\overline{w}(x, t) = \mathbb{E}w(x, t)$ and the covariance function $C_w(x_1, t_1; x_2, t_2)$ at points $(x, t), (x_1, t_1), (x_2, t_2)$ in Q. There are different ways of accomplishing this objective. It is well known that deterministic methods based on the averaging of solutions for different realizations of the random fields f and w_0 turned out to be inefficient. Randomized algorithms of the Monte Carlo method [2] can be employed here as an alternative, randomized methods of solving

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SDEs, in particular [3], which are based on probabilistic representations and need more thorough justification in the considered case of random data. It is known, however, that C_w can be calculated through solving a system of deterministic Cauchy problems with the data determined by C_f, C_{w_0} and C_{f,w_0} [1]. The analogous approach has been employed in [4] as a basis for constructing a Monte Carlo method for the biharmonic equation with random load.

So we have

$$\begin{aligned} &H\overline{w} = \overline{f}, \\ &\overline{w}(x,0) = \overline{w}_0(x), x \in I\!\!R^m, \end{aligned} \tag{3}$$

$$\begin{aligned} &H_{(1)}C_w = S, \\ &C_w(x_1,0;x_2,t_2) = v_0(x_1;x_2,t_2), \quad x_1 \in I\!\!R^m, (x_2,t_2) \in Q, \end{aligned} \tag{4}$$

$$\begin{aligned} &H_{(2)}S = C_f, \\ &S(x_1,t_1;x_2,0) = C_{f,w_0}(x_1,t_1;x_2), \quad x_2 \in I\!\!R^m, (x_1,t_1) \in Q, \end{aligned} \tag{5}$$

$$\begin{aligned} &H_{(2)}v_0 = C_{w_0,f}, \\ &v_0(x_1;x_2,0) = C_{w_0}(x_1;x_2), \quad x_1, x_2 \in I\!\!R^m. \end{aligned} \tag{6}$$

Suppose that $u(x,t) \in L_\infty(Q)$. Then one can seek a solution to (3) ((4), (5), (6), respectively) as a sum of heat potentials, the parametrix Z being their kernel:

$$\begin{aligned} \overline{w}(x,t) = &\int_0^t dt' \int_{I\!\!R^m} dx' Z(x-x',t-t')[-u\cdot\nabla]\overline{w}(x',t') \\ &+ \int_0^t dt' \int_{I\!\!R^m} dx' Z(x-x',t-t')\overline{f}(x't') \\ &+ \int_{I\!\!R^m} dx' Z(x-x',t)\overline{w}_0(x') \\ &\equiv K^{(0)}\overline{w}(x,t) + \overline{F}_0(x,t). \end{aligned} \tag{7}$$

Let $u(x,t) \in C^{1,0}(Q)$ be incompressible, then the Ostrogradsky' formula implies that $\overline{w}(x,t)$ satisfies the integral equation

$$\begin{aligned} \overline{w}(x,t) = &\int_0^t dt' \int_{I\!\!R^m} dx' [u\cdot\nabla] Z(x-x',t-t')\overline{w}(x',t') + \overline{F}_0(x,t) \\ &\equiv K\overline{w}(x,t) + \overline{F}_0(x,t). \end{aligned} \tag{8}$$

The same integral representation is valid for C_w, S and v_0 also.

Theorem 1 [5]. *The Neumann series for* (8) *converges in* $C^{2,1}$ *to the solution of the deterministic Cauchy problem* (3).

Theorem 2. *Covariance function can be calculated as a sum of convergent Neumann series*

$$C_w = (I - K_{(1)})^{-1} F_1, \tag{9}$$

where

$$F_1(x_1, t_1; x_2, t_2) = \int_0^{t_1} dt_1' \int_{\mathbb{R}^m} dx_1' Z(x_1 - x_1', t_1 - t_1') S(x_1', t_1'; x_2, t_2) + \int_{\mathbb{R}^m} dx_1' Z(x_1 - x_1', t_1) v_0(x_1'; x_2, t_2),$$

$$S = (I - K_{(2)})^{-1} F_2,$$

$$F_2(x_1, t_1; x_2, t_2) = \int_0^{t_2} dt_2' \int_{\mathbb{R}^m} dx_2' Z(x_2 - x_2', t_2 - t_2') C_f(x_1, t_1; x_2', t_2') + \int_{\mathbb{R}^m} dx_2' Z(x_2 - x_2', t_2) C_{f,w_0}(x_1, t_1; x_2'),$$

$$v_0 = (I - K_{(2)})^{-1} F_3,$$

$$F_3(x_1, t_1; x_2, t_2) = \int_0^{t_2} dt_2' \int_{\mathbb{R}^m} dx_2' Z(x_2 - x_2', t_2 - t_2') C_{w_0,f}(x_1; x_2', t_2') + \int_{\mathbb{R}^m} dx_2' Z(x_2 - x_2', t_2) C_{w_0}(x_1; x_2').$$

The proof of Theorem 2 is analogous to that of Theorem 1. It is based on the properties of the weakly singular Volterra integral operators $K_{(1)}$ and $K_{(2)}$. □

RANDOM WALKS, MONTE CARLO ESTIMATORS

The uniform convergence of the Neumann series makes it possible to construct unbiased Monte Carlo estimators for $\overline{w}$, C_w and integral functionals of these functions.

Denote by $\{(y_n, \tau_n), n = 0, 1, \ldots\}$ a Markov chain with the transition density $p(y, \tau \to y', \tau')$ consistent with the kernel of the integral operator:

$$p(y,\tau \to y',\tau') = q\,\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)\nu^{1/2}\pi^{m/2}} \cdot \frac{1}{\tau^{1/2}\left(4\nu(\tau-\tau')\right)^{m/2+1}} \times \exp\left(-\frac{|y-y'|^2}{4\nu(\tau-\tau')}\right)\cdot |y-y'|, \tag{10}$$

for $\tau > \tau' > 0$ and zero in all other cases. Here q is the survival probability.

This means that $y_{n+1} = y_n + 2(\nu(\tau_n - \tau_{n+1})\gamma^{(n)}_{(m+1)/2})^{1/2}\omega^{(n)}$, $\tau_{n+1} = (1-\alpha^2)\tau_n$, where α is uniformly distributed in $(0,1)$, $\omega^{(n)}$ is a unit isotropic vector and $\gamma^{(n)}_{(m+1)/2}$ is Γ–distributed with the parameter $(m+1)/2$. Random walk $\{(y_n,\tau_n)\}$ in Q converges in $L_2(\Omega)$ to $(y_\infty, 0)$ and $\{y_n\}$ forms a martingale.

To compute the mean solution at a fixed point (x,t), we put $(y_0,\tau_0) = (x,t)$, construct the Markov chain (y_n,τ_n) in accordance with the transition probability (10) and an adjoint estimator (the collision estimator is used here) as a functional of its path:

$$\xi^*[\overline{w}](x,t) = \sum_{n=0}^{N} Q^*_n(\tau_n \overline{f}(y^*_{n,1},\tau^*_n) + \overline{w}_0(y^*_{n,2})). \tag{11}$$

Here N is a random number of the last point of the path, τ^*_n is uniformly distributed in $(0,\tau_n)$, $y^*_{n,1}$ is of the Gaussian distribution $N(y_n, 2\nu(\tau_n - \tau^*_n)$ (τ^*_n fixed), $y^*_{n,2}$ is of the Gaussian distribution $N(y_n, 2\nu\tau_n)$, and $Q^*_{n+1} = \dfrac{k(y_n,\tau_n;y_{n+1},\tau_{n+1})}{p(y_n,\tau_n \to y_{n+1},\tau_{n+1})}\, Q^*_n = a^*_n Q^*_n$ being random weights,

$$a^*_n = \frac{2}{q}\left(\frac{\tau_n}{\nu}\right)^{1/2}\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\left[-u(y_{n+1},\tau_{n+1})\cdot\omega^{(n)}\right]. \tag{12}$$

By the appropriate choice of the survival probability one can obtain the uniform boundedness of the weight factors a^*_n thus ensuring the unbiasedness and the finiteness of variance of the estimator (11) [5].

One can easily see that the problems (5) and (6) are totally analogous to the Cauchy problem (3). So,

$$\xi^*[S](x_1,t_1;x_2,t_2) = \sum_{n=0}^{N} Q^*_n(\tau_n C_f(x_1,t_1;y^*_{n,1},\tau^*_n) + C_{f,w_0}(x_1,t_1;y^*_{n,2})), \quad (y_0,\tau_0) = (x_2,t_2), \tag{13}$$

$$\xi^*[v_0](x_1;x_2,t_2) = \sum_{n=0}^{N} Q^*_n(\tau_n C_{w_0,f}(x_1;y^*_{n,1},\tau^*_n) + C_{w_0}(x_1;y^*_{n,2})), \quad (y_0,\tau_0) = (x_2,t_2), \tag{14}$$

are unbiased estimators for S and v_0, respectively.

Theorem 3. *The random estimator*

$$\xi^*[C_w](x_1,t_1;x_2,t_2) = \sum_{n=0}^{N} Q^*_n(\tau_n \xi^*[S](y^*_{n,1},\tau^*_n;x_2,t_2) + \xi^*[v_0](y^*_{n,2};x_2,t_2)), \quad (y_0,\tau_0) = (x_1,t_1), \tag{15}$$

is an unbiased estimator for C_w with uniformly bounded variance.

To prove the Theorem suffices it to apply the double randomization principle to the estimator (15) and make use of the properties of the weakly singular Volterra integral operator $K_{(1)}$. $\square$

To reduce the variance of the estimators constructed one can employ the choice of the survival probability q. It can be taken even greater than 1 (thus diminishing weight factors a_n^*) converting $\{(y_n, \tau_n)\}$ into a branching Markov chain.

It is essential to note that the same technique is effective when calculating solutions to the boundary value problems for the equation (1). Potential representation or Green's formula lead to a system of Volterra integral equations with the convergent Neumann series (the choice depends on the kind of the problem). Therefore, Monte Carlo estimators (vector and scalar, collision and absorption, direct and adjoint, local, etc.) can be constructed in these cases also.

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