Monte Carlo solution of a parabolic equation with a random coefficient*

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Рассматривается уравнение параболического типа, в котором коэффициент при решении, правая часть и начальные значения являются случайными функциями. Построены и обоснованы оценки метода Монте-Карло для выборочных значений решения и для некоторых функционалов от него.

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A parabolic equation is considered. Its coefficient in the linear term, the right-hand side, and the initial value of the equation are random functions. Some Monte Carlo estimators for sample values of its solution and for some functionals are constructed and verified.

1. Introduction

In this paper, we consider a multidimensional parabolic equation

$$w_t = \nu \Delta w + cw + f, \tag{1.1}$$

or $H_{x,t}[c]w = f$, where $\nu > 0$ is constant, and coefficient $c(x,t;\omega)$, $(x,t) \in D_T$, $D_T = \mathbb{R}^m \times (0,T)$ is supposed to be a random field with respect to a suitable probability space $(\Omega, \mathcal{A}, P)$. This means that $c(\cdot,\cdot;\omega)$, for each ω fixed, is an element of some functional space X of real-valued functions defined on D_T, and it induces a probability measure μ on a σ-algebra $\mathcal{B}$ of subsets of X.

It is clear that a solution of (1.1) is to be considered in a probabilistic sense. Different approaches can be used here.

The common one is to define a random solution as a random function $w(x,t;\omega)$, which is a classical solution μ-almost surely (a.s.). It means that for almost all fixed ω the function $w(x,t;\omega)$ is continuously differentiable with respect to t, twice continuously differentiable with respect to $x_j, j = 1,\dots,m$, and satisfies equation (1.1). The conditions ensuring the existence of a unique solution of (1.1) (provided the corresponding well-posed boundary or initial value problem is set) are well-known (see, for example, [8, 9]). An essential additional requirement to be fulfilled is also as follows: the random function $w(x,t;\cdot)$ has to be μ-measurable over $(X,\mathcal{B})$ or, in other words, the random variable $w(x,t;\cdot)$ for each fixed $(x,t) \in D_T$ has to be measurable over $(\Omega,\mathcal{A})$.

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Another approach is to consider (1.1) in $L_2(\Omega)$ sense, either in its weak [3] or strong formulations. The latter means that all differential operators in (1.1) are considered in a mean square (m.s.) sense. The mean square approach can be substantiated by the fact that sample values of the random solution and even its point distributions are usually of no special interest. Rather, some integral functionals with respect to the distribution w, such as moments or a characteristic functional, are the purpose of computations.

The major motivation for constructing and studying Monte Carlo methods for random parabolic equations and systems is the apparent inefficiency of conventional computational algorithms when applied immediately to solving initial or boundary value problems for (1.1). It would be most reasonable to reformulate the problem so that exactly the required functional is a solution of some deterministic equation. It is worth noting, however, that the reformulation itself can be a formidable and complicated task.

In this paper, we propose some Monte Carlo estimators for sample values of a random solution itself, for its moments and a characteristic functional. In Section 2 we consider the Cauchy problem and the computational algorithm based on a Wiener integral probabilistic representation for its solution. Section 3 is devoted to reformulation of the problem in the form of integral equations based on various potential representations. Randomization of these integral equations leads to the Monte Carlo algorithms described in Section 4. The results of some numerical experiments are presented and discussed in Section 5.

2. A random Cauchy problem and probabilistic representation

First, consider a random Cauchy problem

$$w(x,0;\omega) = w_0(x;\omega) \tag{2.1}$$

and suppose that equation (1.1) and initial condition (2.1) are to be satisfied with probability one.

One of possible ways to obtain a formal solution of the problem is to use the Feynman-Kac formula. This representation in the form of a Wiener integral, which was first established for the deterministic equation [10, 14, 6], admits generalization to the case of random (and even stochastic) coefficient, the right-hand side and initial values [5, 4, 11]. Thus, as a simple consequence of Theorem 2 [5], we have (f is set to be equal to zero for simplicity of formulation)

Theorem 1. *Let c be: (a) a.s. continuous and uniformly (with respect to ω) bounded above: $c(x,t;\omega) \le c_0$; (b) a.s. uniformly (with respect to ω and t) Hölder continuous in $\mathbb{R}^m$; and let w_0 be a.s. continuous and bounded. Then the Cauchy problem (1.1), (2.1) has the unique random solution given by the following expression:*

$$\begin{aligned} w(x,t;\omega) &= \mathbb{E}\left[w_0(x-z(t);\omega)\exp\left(\int_0^t c(z(\tau)-z(t)+x,\tau;\omega)\,d\tau\right)\Big|\omega\right] \\ &= \mathbb{E}\left[w_0(y(t);\omega)\exp\left(\int_0^t c(y(s),t-s;\omega)\,ds\right)\Big|\omega\right]. \end{aligned} \tag{2.2}$$

Here $z(t) = (2\nu)^{1/2}W(t)$, W is a standard Wiener process, $y(t)$ is the Brownian motion starting from the point x at time $t = 0$. It can be considered as the solution of the stochastic Itô equation $y(t) = x + \int_0^t (2\nu)^{1/2} dW(s)$.

In order to make use of the formula (2.2), one has to construct numerically some approximation for $y(t)$. It is possible, in particular, to use the generalized Euler scheme with a constant step size $\tau = t/N$ (see [17] for motivation):

$$y_{n+1} = y_n + (2\nu\tau)^{1/2}\eta_n,$$

where $y_0 = x$ and $\{\eta_n, n = 0, 1, \ldots\}$ is a sequence of independent random vectors in $\mathbb{R}^m$ with normal distribution $\mathcal{N}(0,1)$ of its independent components [1]. The piecewise linear process $\widetilde{y}(t)$ constructed on the grid of points (y_n, t_n) in $\mathbb{R}^{m+1}$, $t_n = n\tau$, approximates $y(t)$ with an accuracy of order τ [18]. As a consequence of the properties of c and w_0, the functional $F(y)$ in the right-hand side of (2.2) is measurable and bounded. Thus, [2], $w(x, t; \omega)$ is approximated by $F(\widetilde{y})$ with an accuracy of order τt for almost all $\omega \in \Omega$. It is also clear that we have to make use of some numerical scheme in order to compute the integral in (2.2).

Hence, we have

Theorem 2. *At the conditions of Theorem 1, for almost all* ω,

$$w_\tau(x, t; \omega) = w_0(y_N; \omega) \exp\left(\frac{\tau}{2} \sum_{n=0}^{N-1} \left(c(y_n, t_n; \omega) + c(y_{n+1}, t_{n+1}; \omega)\right)\right) \tag{2.3}$$

is a τt*-biased estimator for the solution of the random Cauchy problem* (1.1), (2.1). *The conditional variance of* w_τ *(fixed* ω*) is uniformly bounded in* $\overline{D_T}$.

It has been assumed implicitly in (2.3) that it is possible to use the exact values of the random fields w_0 and c. It is worth noting, however, that it is not a usual case, and some simulation procedure must be involved in the process of computation of the sample values of w_t (see, for example, [13, 15]). The efficiency of such a simulation essentially depends on the statistical characteristics of random parameters and, above all, on their spectral properties.

Bearing in mind the latter speculations, let us set aside the computational cost of the random input data simulation and consider the complexity of estimator (2.3) itself. Denote by V_τ its variance and let V be its upper bound. Hence, in order to attain the order ε of the desired probabilistic accuracy (fixed ω) for the sample value of w_τ, we have to simulate $N_{\text{sample}} = O(\varepsilon^{-2}V)$ independent samples of $w_\tau(x, t; \omega)$ (or, what is equivalent, independent paths of the random walk $\{y_n\}$). It is natural to suppose that the bias of the estimator is of the same order as its probabilistic error. Hence $\tau t = O(\varepsilon)$ and, since $\tau = t/N$, we have $N = O(t^2\varepsilon^{-1})$. It follows that the computational cost of estimator (2.3), which is proportional to NN_{sample}, is of order $\varepsilon^{-3}t^2V$.

3. Integral representations of a solution

3.1. Heat potentials

For every $t - t' > 0$ let

$$Z_0(x - x', t - t') = \Big(4\pi\nu(t - t')\Big)^{-m/2} \exp\left(-\frac{|x - x'|^2}{4\nu(t - t')}\right)$$

be the parametrix, i.e. the fundamental solution of the heat equation corresponding to (1.1). We seek for a solution of random Cauchy problem (1.1), (2.1) in the form of the Neumann series for the integral equation

$$\begin{aligned} w(x,t;\omega) &= \int_0^t dt' \int_{\mathbb{R}^m} dx' Z_0(x-x',t-t')c(x',t';\omega)w(x',t';\omega) + \\ &\quad \int_0^t dt' \int_{\mathbb{R}^m} dx' Z_0(x-x',t-t')f(x',t';\omega) + \int_{\mathbb{R}^m} dx' Z_0(x-x',t)w_0(x';\omega) \\ &\equiv K_{Z_0c}w(x,t;\omega) + F_0(x,t;\omega). \end{aligned} \tag{3.1}$$

It means that

$$w = \sum_{n=0}^{\infty} K^n_{Z_0c}F_0, \tag{3.2}$$

where the limit is considered either a.s. or in the m.s. sense.

Now, we introduce some notation. Let $c_n(X;\omega)$ be the product $\prod\limits_{j=1}^{n} c(x_j,t_j;\omega)$, let C_n be the moment functions $\mathbb{E}(c_n|X)$ $(C_0 \equiv 1)$ of the random field $c(x,t;\omega)$, and let $\overline{C}_n = \mathbb{E}(|c_n|\,|X)$, $\widehat{C}_n = \sup\limits_X |C_n|$. Here $X = \{(x_n,t_n), n = 0,1,\ldots\}$ is some (maybe random) sequence of points in D_T.

The properties of the parametrix [8] make it possible to prove

Theorem 3.

1) *If, for almost all fixed ω, c, f, and w_0 are bounded almost everywhere, the Neumann series in* (3.2) *converges uniformly with respect to x, and t to the a.s. continuous function w.*
2) *If, in addition, c and f are a.s. continuous in D_T and uniformly, with respect to t, a.s. Hölder continuous, w is the solution of integral equation* (3.1)*, a.s. belongs to $C^{2,1}(D_T)$, and satisfies* (1.1) *for almost every fixed ω.*
3) *If w_0 is a.s. continuous, then w a.s. satisfies the initial conditions* (2.1).
4) *If, besides,*
   a) *the moment functions of c satisfy uniformly, with respect to X, the conditions $\overline{C}_n \le t^{-n}n!a(n)$, where $a(n) > 0$ are such that the series $\sum\limits_{n=0}^{\infty} a(n)$ converges,and*
   b) *$\mathbb{E}|f|$ and $\mathbb{E}|w_0|$ are bounded functions,*

   then w, which is the sum of the Neumann series in (3.2)*, has a finite expectation and provides an a.s. solution to the random Cauchy problem.*

Proof. It is clear that the first two conditions of the theorem ensure, for almost all fixed ω, the existence of the unique solution to the Cauchy problem [8, 9]. Therefore, it remains to prove, first, the convergence of the Neumann series in $C(D_T)$ and, second, that the solution of (3.1) satisfies both (1.1) and (2.1).

The first statement immediately follows from the corresponding results for the deterministic case of (3.1) (see [16]).

Actually, if f and w_0 are the elements of $L_\infty(D_T)$ and $L_\infty(\mathbb{R}^m)$, respectively, (fixed ω), then, from the properties of the parametrix [8], F_0 belongs to $C(D_T)$.

Now consider $K_{Z_0 c}$ as an integral operator on this functional space. Since $||K_{Z_0 c}|| \leq ||c||_{L_\infty} ||K_{Z_0}||$ and the properties of the parametrix [19] make it possible to prove by induction that $K^n_{Z_0}(1) = t^n/n!$ then, for almost all fixed ω, the Neumann series (3.2) converges in $C(D_T)$ and $||w|| \leq \exp(||c||t)||F_0||$. (Notice that the right hand side of this inequality depends on ω).

Now let w satisfy (3.1) for almost all $\omega \in \Omega$. The same reasoning as in the deterministic case makes it possible to assert that if w, c, and f are continuous in D_T and uniformly, with respect to t, Hölder continuous, then $w \in C^{2,1}$, and we have

$$H_{x,t}[0]w = cw + f$$

i.e. w satisfies (1.1) for almost all fixed ω. For continuous w_0, the solution of (3.1) also satisfies initial conditions (2.1).

Consider the expectation of w, which is the solution of (3.1). Suppose for simplicity that c and f, w_0 are independent. Then we have

$$\begin{aligned}\mathbb{E}|w| \; &\leq \; \sum_{n=0}^{\infty} \mathbb{E} K^n_{Z_0}\left(|c_n||F_0|\right) = \sum_{n=0}^{\infty} K^n_{Z_0}\left(\mathbb{E}|c_n| \cdot \mathbb{E}|F_0|\right) \leq \sup_{x,t} \mathbb{E}|F_0| \sum_{n=0}^{\infty} K^n_{Z_0}\left(\overline{C}_n\right) \\ &\leq \; (t \sup_{x,t} \mathbb{E}|f| + \sup_{x} \mathbb{E}|w_0|) \sum_{n=0}^{\infty} K^n_{Z_0}(1) \sup_{X} \overline{C}_n .\end{aligned}$$

From the properties of the moment functions (condition 4a of the theorem) and those of the integral operator K_{Z_0}, this series converges. Hence, w defined by (3.2) has a finite expectation, is measurable and, thus, provides a.s. solution to the random Cauchy problem. □

Now consider (3.1) in the m.s. sense and prove the following:

Theorem 4. *Let*

1) *c, f and w_0 have bounded second moments and be m.s. continuous;*
2) *c and f be m.s. Hölder continuous in x, uniformly with respect to t;*
3) *the moment functions of c satisfy the inequalities* **(A)** $\widehat{C}_{2n} \leq (n!)^2 t^{-2n} a^2(n)$, *where* $a(n) > 0$ *are such that the series* $\sum_{n=0}^{\infty} a(n)$ *converges.*

   Then the sum of the series in (3.2) *provides the m.s. solution of the random Cauchy problem* (1.1), (2.1).

Proof. Denote by $\Gamma[u, v]$ the mixed moment function for random fields u and v.

Let w satisfy (3.1) in the m.s. sense. This means that

$$\begin{aligned}\Gamma[w,w](x_1,t_1;x_2,t_2) \; = \; & K^{(1)}_{Z_0} K^{(2)}_{Z_0}(\Gamma[cw,cw] + \Gamma[cw,f] + \Gamma[f,cw] + \Gamma[f,f]) + \\ & K^{(1)}_{Z_0} \int_{\mathbb{R}^m} Z^{(2)}_0 (\Gamma[cw,w_0] + \Gamma[f,w_0]) + \\ & K^{(2)}_{Z_0} \int_{\mathbb{R}^m} Z^{(1)}_0 (\Gamma[w_0,cw] + \Gamma[w_0,f]) + \int_{\mathbb{R}^m} Z^{(1)}_0 \int_{\mathbb{R}^m} Z^{(2)}_0 \Gamma[w_0,w_0], \quad (3.3)\end{aligned}$$

where upper indices denote integration variables and the variables on which the kernel Z_0 depends.

Prove now that w belongs to $L_2(\Omega)$ uniformly with respect to (x,t). For every $(x,t) \in D_T$, we have (as a consequence of the Minkovski inequality)

$$\left\| \sum_{n=0}^{\infty} K^n_{Z_0c} F_0 \right\|_{L_2(\Omega)} (x,t) \leq \sum_{n=0}^{\infty} ||K^n_{Z_0c} F_0||_{L_2(\Omega)}(x,t).$$

Suppose for simplicity that c and F_0 are independent. Then we have by the definition that

$$(||K^n_{Z_0c} F_0||_{L_2(\Omega)}(x,t))^2 = \mathbb{E}(K^n_{Z_0c} F_0)^2(x,t) = K^n_{Z_0}{}' K^n_{Z_0}{}'' \left(c_{2n}\, \mathbb{E}\left(F_0(x'_n, t'_n) F_0(x''_n, t''_n)\right)\right).$$

It follows that w defined by (3.2) satisfies

$$||w||_{L_2(\Omega)}(x,t) \leq \sup_{x,t} ||F_0||_{L_2(\Omega)}(x,t)) \sum_{n=0}^{\infty} K^n_{Z_0}(1) (\widehat{C}_{2n})^{1/2}.$$

Therefore, since $K^n_{Z_0}(1) = t^n/n!$, **(A)** ensures convergence of the series in this relation.

Thus, the random functions c, f, w_0, and w belong to $L_2(\Omega)$ uniformly with respect to (x,t) and, therefore, from the properties of the heat potentials [8], $\Gamma[w,w](x_1,t_1;x_2,t_2)$ is continuous, i.e., w is m.s. continuous. Hence, as a consequence of the first condition and, again, from the properties of the potentials, this moment function is continuously differentiable with respect to x_1 and x_2. Thus, w belongs to $C^{1,0}$ in the m.s. sense and, therefore, from the properties of the potentials, w also belongs to $C^{2,1}$.

It follows that we can apply the operator $H_{x_1,t_1}[0]H_{x_2,t_2}[0]$ to $\Gamma[w,w](x_1,t_1;x_2,t_2)$, and the result will be a continuous function. Next, we make use of (3.3) bearing in mind the continuity properties of the moment functions in its right-hand side. Hence, we have

$$H_{x_1,t_1}[0]H_{x_2,t_2}[0]\Gamma[w,w] = \Gamma[cw,cw] + \Gamma[cw,f] + \Gamma[f,cw] + \Gamma[f,f],$$

which means exactly that w satisfies (1.1) in the m.s. sense. Similarly, from the m.s. continuity and the properties of the heat potentials, the m.s. limit of the solution equals w_0. $\square$

Statement. *The conditions* **(A)** *of Theorem 4 are satisfied if either a) c is bounded almost everywhere in $D_T \times \Omega$ or b) c is a Gaussian random field with bounded second moments.*

Proof. If $|c(x,t,;\omega)| \leq C_{\max}$ then, clearly, $\widehat{C}_{2n} \leq C^{2n}_{\max}$. Thus, we can set $a(n) = C^{2n}_{\max} t^{2n}/(2n)!$ and $\sum_{n=0}^{\infty} a(n) = \sum_{n=0}^{\infty} C^n_{\max} t^n/n! = \exp(C_{\max} t)$.

If c is Gaussian, $|C_1| = |\mathbb{E}c(x,t;\cdot)| \leq M$, $\mathbb{E}(c(x,t;\cdot) - C_1(x,t))^2 \leq \sigma^2$, we have $\widehat{C}_{2n} \leq M\widehat{C}_{2n-1} + (2n-1)\sigma^2 \widehat{C}_{2n-2}$ and, hence, $\widehat{C}_{2n} \leq rs^n n!$ for some constant r and s. It follows that we can set $a^2(n) = r(t^2 s)^n/n!$. $\square$

3.2. Integral representation with a "frozen" coefficient

It is clear that integral representation (3.1) is not the only one that is possible. We can reduce the Cauchy problem to summation of the same Neumann series (3.2) if we seek for a solution either in the form of the heat potential $\int_0^t dt' \int_{\mathbb{R}^m} dx' Z_0(x-x', t-t') \rho(x', t'; \omega)$ with an unknown random density ρ (see, e.g. [12]), or in the form of the sum of potentials

$\int_0^t dt' \int_{\mathbb{R}^m} dx' Z_c(x-x', t-t';\omega) f(x',t';\omega)$ and $\int_{\mathbb{R}^m} dx' Z_c(x-x',t;\omega) w_0(x';\omega)$. Here Z_c is an unknown random fundamental solution of the operator $H[c]$ which is sought for in the form of the Neumann series for an integral equation with the integral operator $K_{Z_0 c}$ [8].

Here we propose to consider one more integral representation based on a technique that may be called "the method of freezing" of the random coefficient c (see [15]). We seek for a solution of the Cauchy problem (1.1), (2.1) in the form of a sum of two potentials

$$w(x,t;\omega) = \int_0^t dt' \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',t';\omega)\rho(x',t';\omega) + \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',0;\omega) w_0(x';\omega). \quad (3.4)$$

Here $Z_c'(x,t;x',t';\omega) = Z_0(x-x',t-t')\exp(c(x',t';\omega)(t-t'))$ is a random kernel, ρ is an unknown potential density and (3.4) is satisfied in an appropriate probabilistic sense.

Suppose that c, w_0, and ρ are continuous and uniformly, with respect to t, Hölder continuous random functions. Let w_0 and ρ be a.s. bounded and c a.s. bounded above (by some random variable from $L_2(\Omega)$). Then, from the properties of the parametrix, the potentials in (3.4) a.s. belong to $C^{2,1}(D_T)$, and we can interchange the order of integration and differentiation. Hence,

$$\begin{aligned} H_{x,t}[0]w = & \lim_{t'\to t-0} \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',t';\omega)\rho(x',t';\omega) + \\ & \int_0^t dt' \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',t';\omega) c(x',t';\omega)\rho(x',t';\omega) + \\ & \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',0;\omega) c(x',0;\omega) w_0(x';\omega). \end{aligned}$$

As a consequence of the properties of the parametrix, we have

$$\begin{aligned} & \lim_{t'\to t-0} \int_{\mathbb{R}^m} dx' Z_c'(x,t;x',t';\omega)\rho(x',t';\omega) \\ & = \lim_{\substack{t'\to t-0 \\ x'\to x}} (\exp(c(x',t';\omega)(t-t'))\rho(x',t';\omega)) = \rho(x,t;\omega) \end{aligned}$$

both a.s. and in the m.s. sense. Thus, since we require that w satisfy (1.1), the density ρ must satisfy the integral equation

$$\begin{aligned} & \rho(x,t;\omega) \\ = & \int_0^t dt' \int_{\mathbb{R}^m} dx' Z_0(x-x',t-t')\exp(c(x',t';\omega)(t-t')) \times [c(x,t;\omega) - c(x',t';\omega)]\,\rho(x',t';\omega) + \\ & \int_{\mathbb{R}^m} dx' Z_0(x-x',t)\exp(c(x',0;\omega)t) \times [c(x,t;\omega) - c(x',0;\omega)]\, w_0(x';\omega) + f(x,t;\omega) \end{aligned} \quad (3.5)$$

or

$$\rho = K_{Z'(c-c')}\rho + F_0'.$$

Theorem 5. *If, for almost all fixed ω, random functions f, w_0, and c are bounded almost everywhere, c and f are a.s. continuous in D_T and uniformly, with respect to t, a.s. Hölder*

continuous, then w, the integral functional of the solution of the integral equation (3.5)*, a.s. belongs to $C^{2,1}(D_T)$ and satisfies* (1.1)*.*
If w_0 is a.s. continuous, then w a.s. satisfies initial conditions (2.1)*.*
If, besides, $\mathbb{E}|w_0|$ and $\mathbb{E}|f|$ are bounded functions, c is a.s. Hölder continuous in D_T, uniformly with respect to ω, and there exists such a constant c^* that $\mathbb{E}\exp(ct) \le \exp(c^* t^2)$, then w defined by (3.4) *provides an a.s. solution to the random Cauchy problem.*

Proof. It is clear that the conditions of the theorem ensure the existence of the unique bounded solution to the Cauchy problem for almost all fixed ω. To prove that w given by (3.4) a.s. coincides with this solution, consider the free term of integral equation (3.5). For almost all ω and almost everywhere in D_T, we have

$$|F_0'|(x,t;\omega) \le 2c_{\max}\exp(c_{\max}t)w_{0,\max} + f_{\max} < \infty$$

for some finite $c_{\max}$, $w_{0,\max}$, and $f_{\max}$ that may depend on ω. It means that $F_0' \in L_\infty$ a.s.

Consider $K_{Z'(c-c')}$ as an operator acting on this functional space. We have $||K_{Z'(c-c')}|| \le 2||c\cdot\exp(ct)||_{L_\infty}\cdot||K_{Z_0}||$ and, therefore, $||K^n_{Z'(c-c')}|| \le (c_{\max}\exp(c_{\max}t))^n t^n/n!$. Hence, the Neumann series in (3.5) converges uniformly in D_T to ρ, an element of $L_\infty(D_T)$. From the properties of the parametrix, the second condition of the theorem ensures the differentiability of ρ and, hence, of w. Then (3.5) means that w defined by (3.4) satisfies (1.1) and the continuity of w_0 ensures that the initial value of w is a.s. equal to this function.

Suppose for simplicity that c, w_0 and f are independent, and consider the expectation of w defined by (3.4), where ρ is a solution of the integral equation (3.5). We have

$$\begin{aligned}\mathbb{E}|w| \le\ & K_{Z_0}\left(\mathbb{E}\left(\exp(c(x_0,t_0;\cdot)(t-t_0))|\rho(x_0,t_0;\cdot)|\right)\right) + \\ & \max_{x',t'}\mathbb{E}\exp(c(x',t';\cdot)t)\max_{x'}\mathbb{E}|w_0(x';\cdot)|.\end{aligned}$$

Considering the first term, we have

$$\begin{aligned}& K_{Z_0}(\mathbb{E}(\exp(c(x_0,t_0;\cdot)(t-t_0))|\rho(x_0,t_0;\cdot)|)) \\ & \quad \le \sum_{n=0}^{\infty} K_{Z_0}(\mathbb{E}(\exp(c(x_0,t_0;\cdot)(t-t_0))|K^n_{Z'(c-c')}F_0'|)) \\ & \quad = \sum_{n=0}^{\infty} K^{n+1}_{Z_0}\left(\mathbb{E}\left|\exp(S_{0,n})\cdot\Pi_n\cdot F_0'\right|\right), \end{aligned} \tag{3.6}$$

where we use the notation

$$S_{0,n} = \sum_{i=0}^{n} c(x_i,t_i;\cdot)(t_{i-1}-t_i),\ t_{-1}\equiv t, \quad \Pi_n = \prod_{j=0}^{n-1}(c(x_i,t_i;\cdot)-c(x_{i+1},t_{i+1};\cdot)).$$

By virtue of Hölder continuity, $|\Pi_n| \le \text{const}\prod\limits_{j=0}^{n-1}|(x_i,t_i)-(x_{i+1},t_{i+1})|^\kappa$. Thus, as a consequence of the last condition of the theorem and the trivial inequality $\sum\limits_{i=-1}^{n-1}(t_i-t_{i+1}) = t-t_n < t$ we have $S_{0,n} \le \exp(c^*t^2)$. Then

$$K^{n+1}_{Z_0}(\mathbb{E}|\exp(S_{0,n})\cdot\Pi_n\cdot F_0'|) \le \exp(c^*t^2)K^{n+1}_{Z_0|\cdot|^\kappa}(\mathbb{E}|f| + \exp(c^*t^2)K_{Z_0|\cdot|^\kappa}\mathbb{E}|w_0|).$$

From the properties of the integral operator $K_{Z_0|\cdot|^\kappa}$ the series in (3.6) converges (see [16, 8]). This means that the expectation of w given by (3.4) is finite, this random function is measurable, and provides an a.s. solution to random Cauchy problem (1.1), (2.1). □

4. Monte Carlo estimators

Convergence of the Neumann series for integral equations (3.1) and (3.4) makes it possible to construct Monte Carlo estimators for w, a solution of random Cauchy problem (1.1), (2.1). In order to do this, define a Markov chain $X = \{(x_n, t_n), n = 0, 1, \ldots\}$ with the state space D_T. It is completely determined by specifying the initial point $(x_0, t_0) = (x, t)$ and the transition density $p(x_n, t_n \to x_{n+1}, t_{n+1}) = \dfrac{q}{t_n} Z_0(x_n - x_{n+1}, t_n - t_{n+1})$. The latter means that t_{n+1} is uniformly distributed on the interval $(0, t_n)$ and $x_{n+1} \in \mathbb{R}^m$ is normally distributed (fixed t_{n+1}): $\mathcal{N}(x_n, 2\nu(t_n - t_{n+1}))$. Thus, t_{n+1} can be simulated in accordance with the formula $t_{n+1} = \alpha t_n$ and x_{n+1} is sampled in accordance with $x_{n+1} = x_n + 2(\nu(t_n - t_{n+1})\gamma_{m/2})^{1/2}\varpi$. Here and below, α are independent sample values of the standard random variable uniformly distributed on $[0, 1]$, $\gamma_{m/2}$ are independent gamma-distributed random variables with the parameter $m/2$, and $\varpi \in \mathbb{R}^m$ are independent isotropic random vectors of unit length.

Next, we sample random functions $c(x, t; \omega)$, $f(x, t; \omega)$, and $w_0(x; \omega)$ in accordance with some simulation procedure, and construct a Monte Carlo adjoint collision estimator [7]:

$$\xi^*[w](x, t; \omega) = \sum_{n=0}^{N} Q_n^*[c]\, \xi[F_0](x_n, t_n; \omega), \tag{4.1}$$

or, in the recurrence form,

$$\xi^*[w](x_n, t_n; \omega) = s_n \frac{t_n}{q} c(x_{n+1}, t_{n+1}; \omega)\xi^*[w](x_{n+1}, t_{n+1}; \omega) + \xi[F_0](x_n, t_n; \omega). \tag{4.2}$$

Here

$$Q_{n+1}^*[c] = Q_n^*[c]\, \frac{k(x_n, t_n; x_{n+1}, t_{n+1})}{p(x_n, t_n \to x_{n+1}, t_{n+1})} = Q_n^*[c]\, \frac{t_n}{q}\, c(x_{n+1}, t_{n+1}; \omega), \quad Q_0^*[c] = 1, q$$

is the survival probability on the transition $x_n, t_n \to x_{n+1}, t_{n+1}$ (it may depend on n), $\{s_n, n = 0, 1, \ldots\}$ is a sequence of independent random variables such that $s_n = 1$ with a probability q and $s_n = 0$ with a complementary probability $1 - q$. N is a random number of terms in a sample path of the Markov chain X.

Note that, in accordance with the double randomization principle, we use unbiased estimators $\xi[F_0]$ instead of the exact values of the integrals constituting F_0. This is natural in this context, since there is no possible way to obtain analytical values of these integrals. Here we have

$$\xi[F_0](x_n, t_n; \omega) = t_n f(x_{n,1}^*, t_n^*; \omega) + w_0(x_{n,2}^*; \omega),$$

where $x_{n,2}^*$ has a normal distribution $\mathcal{N}(x_n, 2\nu t_n)$, $x_{n,1}^*$ has a conditional (t_n^* fixed) normal distribution $\mathcal{N}(x_n, 2\nu(t_n - t_n^*))$, and t_n^* is uniformly distributed over $(0, t_n)$ (see [16]).

Theorem 6. *At the assumptions of Theorem 3, we have*

1) *if only the first condition is valid, i.e. for almost every fixed $\omega \in \Omega$ c, f and w_0 are bounded almost everywhere in D_T, then a.s. there exists $\mathbb{E}(\xi^*[w]|\omega)$, which is equal to the sum of the Neumann series in* (3.2);
2) $\mathbb{E}(\xi^*[w]|\omega)$ *provides an a.s. solution to the random Cauchy problem* (1.1), (2.1), *and it has a finite conditional variance;*
3) *for every finite $t > 0$, $\xi^*[w](x, t; \omega)$ is the unbiased estimator for the mean value of the solution $\overline{w}(x, t) \equiv \mathbb{E}w(x, t; \cdot)$;*

4) *if, besides, f and w_0 have bounded second moments and the moment functions of c satisfy the conditions $\overline{C}_n \le n!!q^{n/2}t^{-n}a(n)$ for some convergent series $\sum a(n)$, then its variance is finite.*

Proof. Introduce a random variable $\Delta_n(X)$, which is equal to one if $n \le N$ and equal to zero otherwise. From the general theory of Monte Carlo estimators [7] it follows that

$$\mathbb{E}(Q_n^*[1]\Delta_n(X)) = K_{Z_0}^n(1)$$

and it is equal to $t^n/n!$ as we have already seen. Hence, we can write

$$\begin{aligned}\mathbb{E}(|\xi^*[w]|\,|\omega) &= \mathbb{E}\left(\left|\sum_{n=0}^{\infty} Q_n^*[1]\, c_n(X;\omega)\xi[F_0](x_n,t_n;\omega)\Delta_n(X)\right| \,\middle|\, \omega\right)\\ &\le \mathbb{E}\left(\sum_{n=0}^{\infty} Q_n^*[1]\,|c_n(X;\omega)||\xi[F_0](x_n,t_n;\omega)|\Delta_n(X)\,\middle|\,\omega\right)\\ &= \sum_{n=0}^{\infty}\mathbb{E}\left(Q_n^*[|c|]|\xi[F_0]|\Delta_n(X)\,\middle|\,\omega\right) = \sum_{n=0}^{\infty} K_{Z_0|c|}^n\left(\mathbb{E}(|\xi[F_0]|\,\middle|\,X,\omega)\right).\end{aligned} \tag{4.3}$$

Since, almost surely, $|\xi[F_0](x_n,t_n;\omega)| \le tf_{\max}(\omega) + w_{0,\max}(\omega) \equiv F_{\max}$, we have

$$\mathbb{E}(|\xi^*[w]|\,|\omega) \le F_{\max}\sum_{n=0}^{\infty} K_{Z_0|c|}^n(1) \le F_{\max}\sum_{n=0}^{\infty}(c_{\max}(\omega)t)^n/n! = F_{\max}\exp(c_{\max}t) < \infty$$

for almost all fixed ω. It follows that, for almost all fixed ω, we can expand the series in (4.1) to infinity by introducing the multiplier $\Delta_n(X)$, and average this series term-by-term. Thus, we conclude that

$$\mathbb{E}(\xi^*[w](x,t;\omega)|\omega) = \sum_{n=0}^{\infty} K_{Z_0 c}^n F_0(x,t;\omega), \tag{4.4}$$

and from the first condition of Theorem 3, the Neumann series in the right-hand side converges.

It has been established already that the conditions of Theorem 3 ensure that the sum of this series provides the solution to the random Cauchy problem. Relation (4.4) means that the same is true for the conditional estimator $\mathbb{E}\left(|\xi^*[w]|\,|\omega\right)$.

Suppose for simplicity that c and f, w_0 are independent, and consider the mean value of the estimator (4.1) bearing in mind the estimate (4.3).] Thus, we have

$$\begin{aligned}\mathbb{E}|\xi^*| &= \mathbb{E}\mathbb{E}(|\xi^*|\,|\omega) \le \sum_{n=0}^{\infty}\mathbb{E}K_{Z_0|c|}^n\left(\mathbb{E}(|\xi[F_0]|\,\middle|\,X,\omega)\right) \le \sum_{n=0}^{\infty} K_{Z_0}^n\left(\overline{C}_n\mathbb{E}(K_{Z_0}|f| + \int Z_0|w_0|)\right)\\ &\le (t\sup_{x,t}\mathbb{E}|f| + \sup_x \mathbb{E}|w_0|)\sum_{n=0}^{\infty} K_{Z_0}^n(1)\sup_X \overline{C}_n < \infty,\end{aligned}$$

since, almost surely, [16]

$$\begin{aligned}\mathbb{E}(|\xi[F_0](x_n,t_n;\omega)|\,|X,\omega) &= \mathbb{E}(|t_n f(x_{n,1}^*,t_n^*;\omega) + w_0(x_{n,2}^*;\omega)|\,|X,\omega) \le \mathbb{E}(t_n|f| + |w_0|\,|X,\omega)\\ &= K_{Z_0}|f| + \int_{\mathbb{R}^m} Z_0|w_0|.\end{aligned}$$

It follows that the expectation of $\xi^*[w]$ is equal to the mean value of the Neumann series (3.2), thus providing an unbiased estimator for the mean solution to the random Cauchy problem.

It remains to prove that both $\mathbb{E}(\xi^*[w](x,t;\omega)|\omega)$ and $\xi^*[w]$ have finite variances.

Consider the recurrence representation (4.2) for the estimator and write down its second conditional moment for almost all fixed ω:

$$\begin{aligned}\mathbb{E}(\xi^{*2}[w]|\omega) &= K_{Z_0^2c^2/p}[\mathbb{E}(\xi^{*2}[w]|\omega)] + \mathbb{E}(\xi^2[F_0]|\omega) + 2F_0K_{Z_0c}w \\ &= \sum_{n=0}^{\infty} K^n_{Z_0^2c^2/p}[\mathbb{E}(\xi^2[F_0]|\omega) - F_0^2 + 2F_0\mathbb{E}(\xi^*[w]|\omega)].\end{aligned}$$

Consider $\mathbb{E}(\xi^2[F_0]|\omega)$. It is clear that the simulation of auxiliary points for this estimator depends on our choice, and we can sample them independently. It follows that

$$\mathbb{E}(\xi^2[F_0](x,t;\omega)|\omega) = tK_{Z_0}f^2 + 2K_{Z_0}f\int\limits_{\mathbb{R}^m} Z_0w_0 + \int\limits_{\mathbb{R}^m} Z_0w_0^2.$$

Thus, since c, f, and w_0 are bounded for a fixed ω, the term in brackets is also bounded. Therefore, to establish the finiteness of $\mathbb{E}(\xi^{*2}[w]|\omega)$ it is sufficient to prove convergence of the Neumann series for $K_{k^2/p} = K_{Z_0^2c^2/p} = K_{tZ_0c^2/q}$. Consider the integral operator with the kernel tZ_0. From the properties of the parametrix Z_0, $K^n_{tZ_0}(1) = t^{2n}2^{-n}/n!$. Thus, we have

$$\mathbb{E}(\xi^2[F_0]|\omega) \le \mathrm{const}(\omega)\sum_{n=0}^{\infty}(c_{\max}t)^{2n}(2q)^{-n}/n! < \infty.$$

Notice that for random c the term in brackets is correlated with the kernel of the integral operator. Thus, to estimate the absolute second moment we have to take into account this relation. We obtain

$$\mathbb{E}\xi^{*2}[w] \le \sup_{x,t}\mathbb{E}\left(\xi^2[F_0] - F_0^2\right)\sum_{n=0}^{\infty} q^{-n}\overline{C}_{2n}K^n_{tZ_0}(1) + 2\sup_{x,t}\mathbb{E}F_0^2\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} q^{-n}\overline{C}_{2n+j}K^n_{tZ_0}K^j_{Z_0}(1).$$

It can be easily obtained that

$$K^n_{tZ_0}K^j_{Z_0}(1) = t^{2n+j}(j+1)(j+3)\cdots(j+2n-1)/(2n+j)!.$$

Since it is obvious that

$$(2n)!!\,2^{-n}/n! < 1 \quad \text{and} \quad (2n+j)!!\,(j+1)(j+3)\cdots(j+2n-1)/(2n+j)! < 1/j!!,$$

we have

$$\mathbb{E}\xi^{*2}[w] \le \sup_{x,t}\mathbb{E}\left(\xi^2[F_0] - F_0^2\right)\sum_{n=0}^{\infty} a(2n) + 2\sup_{x,t}\mathbb{E}F_0^2\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} q^{j/2}a(2n+j)/j!!.$$

This means that the second moment of $\xi^*[w]$ is bounded and its variance is finite. □

It is clear that the transition density of the Markov chain X defined in this section is consistent with the kernel of the integral operator in (3.5). Thus, it can be used to construct a Monte Carlo estimator based on the representation (3.4) with a "frozen" coefficient. Namely, we sample the random functions c, f, and w_0 thus fixing ω, and set

$$\xi_f^*[w](x,t;\omega) = t\,\exp(c(x_1,t_1;\omega)(t-t_1))\xi[\rho](x_1,t_1;\omega) + \exp(c(x^*,0;\omega)\,t)w_0(x^*;\omega), \quad (4.5)$$

where the estimator for ρ is determined by the recurrence (for brevity, here we omit the dependence on the random parameter)

$$\begin{aligned}\xi[\rho](x_n,t_n) \;=\; & \xi[\rho](x_{n+1},t_{n+1}) \times \frac{t_n}{q}\exp(c(x_{n+1},t_{n+1})(t_n-t_{n+1}))[c(x_n,t_n)-c(x_{n+1},t_{n+1})] + \\ & \exp(c(x_n^*,0)\,t_n)[c(x_n,t_n)-c(x_n^*,0)]w_0(x_n^*) + f(x_n,t_n). \quad (4.6)\end{aligned}$$

Theorem 7. *At the assumptions of Theorem 5, the conditional estimator $\mathbb{E}(\xi_f^*[w](x,t;\omega)|\omega)$ provides a.s. solution to the random Cauchy problem and has finite conditional variance.*

At the last conditions of Theorem 5, $\xi_f^*[w]$ defined by (4.5) is an unbiased estimator for the mean value of the solution at the point $(x,t)\in D_T$ and has finite variance.

Proof. To prove the first part of the theorem, it is sufficient to repeat, almost literally, the speculations of Theorem 6. Thus, it remains to establish that the variance of $\xi_f^*[w]$ is finite or, what is sufficient, that $\xi[\rho]$ has a bounded second moment.

Denote by Ψ the conditional (fixed ω) second moment of $\xi[\rho]$. From (4.6) it follows that

$$\Psi = K_{Z_0 e(c)}\Psi + F_0'(\rho - F_0'), \quad (4.7)$$

where

$$e(c) = \frac{t_n}{q}\exp(2c(x_{n+1},t_{n+1})(t_n-t_{n+1}))[c(x_n,t_n)-c(x_{n+1},t_{n+1})]^2.$$

Suppose that the conditions that ensure absolute convergence of the Neumann series for the integral operator $K_{Z_0 e(c)}$ are satisfied. Then, we can write explicitly

$$\begin{aligned}\Psi \;&=\; \sum_{i=0}^{\infty} K^i_{Z_0 e(c)}\left(F_0'\sum_{n=1}^{\infty}K^n_{Z'(c-c')}F_0'\right) \\ &= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}K^i_{Z_0 e(c)}\left(F_0'K^n_{Z'(c-c')}F_0'\right) + \sum_{n=1}^{\infty}F_0'K^n_{Z'(c-c')}F_0'.\end{aligned}$$

In our notation,

$$K^i_{Z_0 e(c)}\left(F_0'K^n_{Z'(c-c')}F_0'\right) = K^{i+n}_{Z_0}\left(\exp(2S_{1,i}+S_{1,n})\Pi_i^2\Pi_n F_0'(x_i,t_i;\cdot)F_0'(x_{i+n},t_{i+n};\cdot)\right),$$

and, by virtue of the properties of the functions S and Π, we come to summation of the series for the integral operator $K_{Z_0|\cdot|^\kappa}$ (see the proof of Theorem 5). The convergence of this series means that the conditional variance of the estimator (4.5) is finite. It is clear, however, that the convergence remains valid if we take the expectations of absolute values of terms in the Neumann series for the integral operator $K_{Z_0|\cdot|^\kappa}$. Therefore, the absolute variance of the estimator is also finite. □

Remark. The computational cost of estimators (4.1) and (4.5) is of order $\varepsilon^{-2}V$, where V is the upper bound of the variance and may depend on t.

5. Results of computations and conclusions

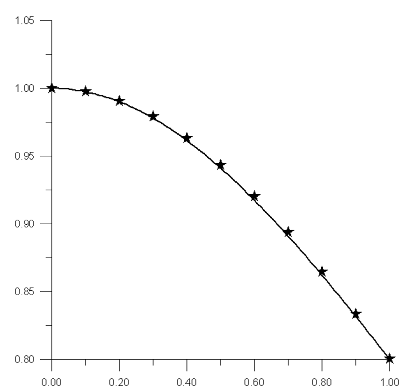
In order to avoid problems that could be caused by simulation of random fields in spaces of higher dimensions, we choose to consider, as a numerical example, the solution of equation (1.1) in the one-dimensional case. We set the test solution to be $\cos(\lambda x t)$, where λ is an exponentially distributed random variable. In this case, $f(x, t; \omega) = -\lambda x \sin(\lambda x t)$, and the initial values are constant and equal to one. Let $\nu = 1$, then $c(x, t; \omega) = (\lambda t)^2$. This means that to obtain sample values of the random fields f and c, we may use the simple simulation formula $\lambda = -\log \alpha$, and we do not need to apply randomized spectral models here [13].

Computational experiments made it clear that with this simulation procedure estimators (4.1) and (4.5) are equivalent, and so the computational results presented are for the estimator (4.1) only. It is essential to note that the randomized Markov chain cut-off in accordance with the chain condition (with probability $1-q$ on every step) is not the only algorithm that is appropriate. Namely, we set q equal to one and terminate the calculations when the weight Q_n^* becomes less than a prescribed value ε.

It was most interesting to estimate the second moment function for the solution. In the test considered it is equal to

$$B(x_1, t_1; x_2, t_2) = \frac{1 + x_1 t_1 + x_2 t_2}{(1 + |x_1 t_1 - x_2 t_2|^2)(1 + |x_1 t_1 + x_2 t_2|^2)}.$$

To obtain an estimator for B, we use two independent estimators (4.1) constructed for the same sample values of the random parameters of the problem and based on the same Markov chain. The figure gives the results of computations (stars) compared to the exact values of $B(0, 0.5; x, 0.5)$.



Notice that the coefficient c in this example does not satisfy the conditions of Theorem 2. This means that estimator (2.3) based on the probabilistic representation is not applicable here. The conditions of Theorem 3 are satisfied only for $t < 1$, and the results of computations are in good agreement with the theoretical curve in this case, and this is clearly seen from the figure. Notice that the applicability of estimator (4.1) strongly depends on the properties of c, and its variance may rapidly grow with t. Hence, some additional efforts are needed in order to apply the constructed estimators for large times.

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